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# Nonlinear Physics

*The three problems are independent. Duration of the examination 4 hours.*

## 1 Diffusion and self-similarity

Let $t \in \mathbb{R}$, $x \in \mathbb{R}$, $y \in \mathbb{R}$, $r = \sqrt{x^2 + y^2}$. Let $\rho = \rho(t, r)$ be the positive solution (density) of the following nonlinear diffusion equation reads:

$$\partial_t \rho = D\Delta\rho - |\rho|^p \rho \tag{1}$$

where $D$ is the diffusion coefficient and $p$ a real positive number and $\Delta$ is the Laplacian.
Show that this equation admits a unique self-similar solution and find the differential equation satisfied by the self-similar solution.

*Hint*: The Laplacian of a function $\Phi(r, \theta)$ in polar coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi(r,\theta)}{\partial \theta^2}. \tag{2}$$

## 2 The FitzHugh-Nagumo model

Let $t \in \mathbb{R}$, $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $(u, v) = (u(t, x, y), v(t, x, y))$ a positive solution of the FitzHugh-Nagumo (FHN) model:

$$\begin{cases} \partial_t u = \Delta u + u - u^3 - v \\ \\ \partial_t v = \delta\Delta v + \alpha(u - a_1 v - a_0) \end{cases}$$

where $a_0$, $a_1$, $\alpha$, and $\delta$ are positive parameters ($a_0$ may be null).
The FHN model has been proposed as a simplified description of spike generation in axons of nerve cells. For us, it represents a canonical model of pattern formation, inspired by reaction diffusion equation.

**1)** Let $(U_0, V_0)$ be a *stationary and homogeneous* solution, which is the intersection of two curves in the diagram $V_0$ versus $U_0$. Show that there exists 3 possible scenarios. Explain why $u$ may be considered as an activator and $v$

as an inhibitor.

**Assume now that** $a_0 = \delta = 0$.

**2)** Show that, depending on the magnitude of the parameter $a_1$, it may exist either one couple or 2 couples of positive homogeneous solutions and find them.

**3)** Assume that $a_1 \leq 1$. Study the stability of the dynamical system FNH under the following expansion:

$$u = U_0 + \epsilon\, e^{\Omega t} \text{ and } v = V_0 + \epsilon\, c\, e^{\Omega t} \tag{3}$$

where $\epsilon$ is a small parameter and $c$ a real constant.

**Assume now that** $a_0 = \delta = 0$ **and** $a_1 \leq 1$.

**4)** Study the stability of the FNH system for perturbations depending on both time and space. Calculate the growth rate $\Omega(k)$, where $k$ is the wavenumber. Show that this system exhibits a Hopf bifurcation at $\alpha_h = 1/a_1$ and give the frequency $\omega_0$ of the oscillations.

**5)** In the vicinity of the Hopf bifurcation, we define the following small positive parameter:

$$\lambda = (\alpha_h - \alpha)/\alpha_h$$

and we consider the following expansion:

$$\begin{cases} u = U_0 + \lambda^{1/2} u_1 + \lambda u_2 + \lambda^{3/2} u_3 + ... \\ \\ v = V_0 + \lambda^{1/2} v_1 + \lambda v_2 + \lambda^{3/2} u_3 + ... \end{cases}$$

where

$$\begin{cases} u_1 = A(X, Y, T) e^{i\omega_0 t} + C.C. \\ \\ v_1 = cA(X, Y, T) e^{i\omega_0 t} + C.C. \end{cases}$$

with $X = \lambda^{1/2} x$, $Y = \lambda^{1/2} y$ and $T = \lambda t$. Here the aim is to find a normal form for $A$ by multiscale analysis.
a) Show that the multiscale analysis for the Hopf bifurcation implies

$$\partial_x \to \lambda^{1/2} \partial_X, \;\; \partial_y \to \lambda^{1/2} \partial_Y, \;\; \partial_t \to \partial_t + \lambda \partial_T \tag{4}$$

b) Show that

$$\mathcal{L}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = 0, \quad \text{with} \quad \mathcal{L} = \begin{pmatrix} \partial_t - 1 & 1 \\ -\alpha_h & \partial_t + 1 \end{pmatrix} \tag{5}$$

Recover the values of $c$ and $\omega_0$.

c) Show that $u_2 = v_2 = 0$ and that

$$\mathcal{L}\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} \mathcal{M} \\ \mathcal{N} \end{pmatrix} e^{i\omega_0 t} + \begin{pmatrix} -A^3 \\ 0 \end{pmatrix} e^{3i\omega_0 t} + C.C. \tag{6}$$

where $\mathcal{M} = -\partial_T A + \partial^2_{X^2} A + \partial^2_{Y^2} A - 3|A|^2 A$ and $\mathcal{N} = -c\partial_T A + (c - \alpha_h)A$.

d) Let $c^*$ be the complex conjugate of $c$. Show by integration by part that, forall $(u_3, v_3)$, it holds

$$\int_0^{2\pi/\omega_0} (-c^*, 1) e^{-i\omega_0 t} \mathcal{L}\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} dt = 0. \tag{7}$$

Note that this equality allows us to eliminate the secular term.
e) Prove that

$$(-c^*, 1)\begin{pmatrix} \mathcal{M} \\ \mathcal{N} \end{pmatrix} = 0 \tag{8}$$

and find the normal form for $A(X, Y, T)$ (this equation is a Landau-Ginzburg equation).

# 3 Lifting the upper plate of an Hele-Shaw cell

A circular disc of a viscous fluid is put in a Hele-Shaw cell of thickness $b(t)$. The upper plate of the cell is lifted with a velocity $\dot{b}$. The viscosity of the fluid is $\mu$, the surface tension between the fluid and the air is $\gamma$.

**1)** Show by means of the mass conservation law that the horizontal velocity $\vec{v}$ satisfies:

$$\vec{\nabla}.\vec{v} = -\dot{b} \tag{9}$$

where $\vec{\nabla}.\vec{v}$ denotes the divergence of $\vec{v}$ in 2 dimensions with variables $(x, y)$.

**2)** Recover the Darcy law assuming that $b$ remains small. Give the equation for the pressure $P$ and the averaged velocity potential $\Phi(r, \theta)$. Solve these equations assuming that the interface is circular at any time with radius

$R_0(t)$.
*Hint*: The Laplacian of a function $\Phi(r,\theta)$ in cylindrical coordinate is

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\Phi(r,\theta)}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi(r,\theta)}{\partial\theta^2} \tag{10}$$

**3)** Do you think that this experiment create a viscous fingering instability? (Explain your response)
In order to study the existence of a possible instability, we assume that the interface radius will evolve as follows:

$$R(t)=R_0(t)+\epsilon_n(t)\cos n\theta \tag{11}$$

under the assumption that the variation of $R_0(t)$ is slow with respect to $\epsilon_n(t)$. How the pressure field or the velocity potential is modified in this case due to the interface ondulation?

**4)** Use the Laplace law to fix the value of pressure at the interface up to linear orderin $\epsilon_n$. The curvature $\mathcal{C}$ in the polar coordinates reads:

$$\mathcal{C}=\frac{R^2+2RR_\theta-RR_{\theta\theta}}{(R^2+R_\theta^2)^{3/2}} \tag{12}$$

where $R_\theta=dR/d\theta$ and $R_{\theta\theta}=d^2R/d\theta^2$.

**5)** Calculate the growth rate $\Omega_n$ and the "expected" mode $n$ of destabilization.

**6)** It is possible to write a Rayleighian $\mathcal{R}$ for the viscous fingering problem as follows:

$$\mathcal{R}=\gamma\oint_{\partial D}\mathcal{C}(\vec{V}.\vec{N})dl+\frac{6\mu}{b^2}\iint_D V^2dS+\iint_D P(\vec{\nabla}.\vec{V}+\dot{b})dS \tag{13}$$

Explain the physical meaning of each integral. Show that this functional allows to recover all the equations of the problem: Volume and interface.