Nonlinear physics 5

December 6, 2016

I. REVERSIBLE SYSTEMS

One considers the following dynamical system

$$\frac{d^2x}{dt^2} = ax - bx^3 + cx\frac{dx}{dt}, \tag{1}$$

with $a, b > 0$ and $x(t)$ real.

1. Show that one can take $c > 0$ without loss of generality.

2. We take $c > 0$; show that the system can be put in the form

$$\ddot{x} = x - x^3 + \nu x\dot{x}, \tag{2}$$

with $\nu > 0$. ($\dot{x}$ stands for the derivative of x with respect to t).

3. What are the stationary solutions of equation (2)?

4. Perform the linear stability analysis of these stationary solutions. Discuss the different signs of ν.

5. One notices that the eigenvalues or growth-rates s obtained in the linear stability analysis, are such that if s is an eigenvalue, then $-s$ is also an eigenvalue. What is the invariance of equation (2) that explains this property ? If s is an eigenvalue, then its complex conjugate, $\overline{s}$, is also an eigenvalue. Why ?

6. One takes $\nu = 0$. Draw the trajectories of the dynamical system (2) in the phase space $(x, \dot{x})$.

7. Show that a necessary condition for a periodic solution of equation (2) is $\langle x^4 \rangle > \langle x^2 \rangle$ where $\langle . \rangle$ stands for the mean value on one oscillation period.

8. One assumes ν small. Draw the trajectories of the dynamical system (2) in the phase space $(x, \dot{x})$. In particular, look for solutions that connects the stationary solutions found at question 3. (Hint: use the invariance of equation (2) in the transformation $t \to -t$ and $x \to -x$.

II. PARAMETRIC AMPLIFICATION. THE HARMONIC CASE

Note that this is different from the subharmonic case, already considered during the lectures.

We consider a pendulum with eigenfrequency ω_0, parametrically forced at frequency Ω. The angle $x(t)$ of the pendulum from the vertical direction, obeys the equation :

$$\frac{d^2x}{dt^2} + \nu \frac{dx}{dt} + \omega_0^2 (1 + f \cos \Omega t) \sin x = 0. \tag{3}$$

One considers parametric forcing at a frequency comparable to the eigenfrequency, $\Omega = \omega_0 + \delta$ ($|\delta| \ll \omega_0$).

Ones looks for a solution $x(t)$ in the form :

$$x(t) \propto A(T) \exp i\Omega t + \bar{A}(T) \exp -i\Omega t + \cdots \tag{4}$$

We recall that if $\nu = 0$ and $f = 0$, the amplitude equation is of the form

$$\frac{dA}{dT} = i\alpha A + i\beta A^2 \overline{A} + \cdots \tag{5}$$

When $f \neq 0$, one tries to guess the new terms of the form $a_{m,n}(f) A^m \overline{A}^n$ that are allowed compared to the previous amplitude equation. One assumes that f being small, $a_{m,n}(f)$ can be expanded in series of successive powers of f.

1. Using an appropriate transformation that leaves equation (3) invariant, and taking into account (4), show that these new terms should be invariant under the transformation $f \to -f$, $A \to -A$. Write the form of the amplitude equation when $\nu \neq 0$ and $f \neq 0$ keeping only the leading order terms.

2. The answer to the previous question leads to the following choice of scalings :

$x(t) = \varepsilon\, v(t,T) = \varepsilon[v_0(t,T) + \varepsilon\, v_1(t,T) + \varepsilon^2\, v_2(t,T) + \cdots]$

$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial T}$

$\delta = \varepsilon^2 \Delta$

$f = \varepsilon F$

$\nu = \varepsilon^2 \Gamma$

where Δ, F, Λ are of order 1 and $\varepsilon \ll 1$.

Do you agree? Why?

3. Perform the asymptotic expansion up to order ε^2 and get the amplitude equation. Find the value of f for which the solution $x = 0$ of (3) becomes unstable. Does the pendulum oscillate symmetrically with respect to $x = 0$? Discuss this last result.

III. NONLINEAR SCHRÖDINGER AND KORTEWEG-DE VRIES EQUATIONS

We start from the nonlinear Schrödinger equation,

$$\frac{\partial A}{\partial t} = i\frac{\partial^2 A}{\partial x^2} - i \mid A \mid^2 A, \tag{6}$$

that governs the complex amplitude $A(x,t)$ of a wave train in a nonlinear and dispersive medium. The coefficients have been taken equal to 1 using an appropriate scalings of time and space and their signs are such that a monochromatic plane wave is stable.

This equation has solutions $A = Q \exp i\Omega t$ with $\Omega = -Q^2$, that represent a family of monochromatic plane waves of constant amplitude. We will try to find the equation governing a small perturbation localized in space. We thus write

$$A(x,t) = [Q + r(x,t)] \exp i[\Omega t + \phi(x,t)].$$

1. Write the governing equations for $\partial r/\partial t$ and $\partial\phi/\partial t$.

2. We look for slowly varying solutions in the form

$$r = r(\xi, \tau), \tag{7}$$

$$\phi = \phi(\xi, \tau), \tag{8}$$

with $\xi = \epsilon(x - ct)$, $\tau = \epsilon^3 t$. We assume that r and ϕ are small perturbations and we expand them in series of a small parameter ϵ,

$$r = \epsilon^2 r_0 + \epsilon^4 r_1 + \cdots, \tag{9}$$

$$\phi = \epsilon\phi_0 + \epsilon^3\phi_1 + \cdots, \tag{10}$$

Using the evolution equations for r and ϕ, show that we get to leading order in ϵ

$$c = \sqrt{2}Q, \tag{11}$$

$$r_0 = \frac{1}{\sqrt{2}}\frac{\partial\phi_0}{\partial\xi}. \tag{12}$$

What is c ?

3. Show that at the next order in ϵ,

$$\frac{\partial r_0}{\partial\tau} + 6\sqrt{2}\, r_0\frac{\partial r_0}{\partial\xi} + \frac{1}{\sqrt{2}Q}\,\frac{\partial^3 r_0}{\partial\xi^3} = 0. \tag{13}$$

4. Show that the previous equation can be put in the form of the Korteweg-de Vries equation

$$\frac{\partial u}{\partial T} + u\frac{\partial u}{\partial X} + \frac{\partial^3 u}{\partial X^3} = 0. \tag{14}$$

5. Find solutions of the Korteweg-de Vries equation in the form $u(X,T) = f(X - CT)$ such that f and its derivatives vanish when the argument of f goes to $\pm\infty$.

IV. DYNAMICS OF ISING WALLS

We consider a simple model of a one-dimensional magnet with a magnetization $M(x,t)$ (positive or negative) along the y-axis (see figure 1).

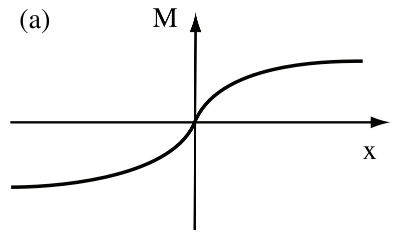

FIG. 1: A domain-wall in an infinite medium

We assume that the evolution of M is

$$\frac{\partial M}{\partial t} = \mu M - \gamma M^3 + {\xi_0}^2 \frac{\partial^2 M}{\partial x^2}, \tag{15}$$

where μ is the relative departure from the Curie temperature T_c, $\mu = (T_c - T)/T_c$, and γ, ${\xi_0}^2$ are real positive constants.

1. We first consider spatially homogeneous solutions of (15). Plot a bifurcation diagram with the different possible stationary solutions as functions of μ and discuss their stability. Describe what happens for $\mu = 0$.

2. Show that for $\mu > 0$, we can consider

$$\frac{\partial M}{\partial t} = 2M(1 - M^2) + \frac{\partial^2 M}{\partial x^2}. \tag{16}$$

without loss of generality compared to equation (15).

3. We consider x-dependent stationary solutions of equation (16). Show that they are related to the motion of a particle of mass unity in a potential $V(M)$ and give the expression of $V(M)$. Plot the phase space $(M, dM/dx)$. Show that there exists an infinite number of solutions periodic in x and two non-periodic limit solutions. Give the qualitative plot of these solutions versus x. Can you find their analytic expression?

4. Using a method described during the lectures, show that all the periodic solutions are unstable and that the limit solutions are stable.

5. We consider the initial condition displayed in figure 2. We assume that the Ising walls (i.e. régions in the vicinity of x_i where the magnetization changes sign) are far enough one from the other such that the shape of the wall approximately corresponds to one of the limit solutions found above. What is the condition on the distance L between two successive walls such that this approximation is correct? Use the notations of equation (15) to answer this question.

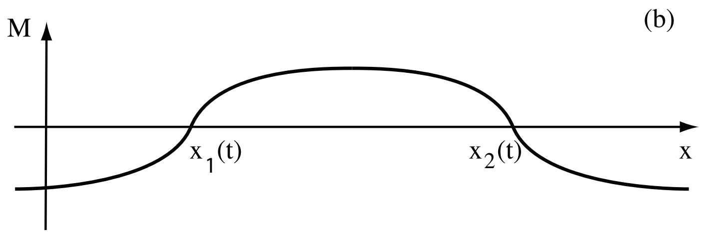

FIG. 2: Two interacting domain-walls

6. Using arguments already considered in question 4, explain in one sentence why the initial condition displayed in figure 2 is unstable. We want to show this directly by studying the dynamics of the Ising walls as time evolves. We first consider the case of two walls displayed in figure 2.

7. Show that one can look for a solution in the form

$$M(x,t) = M_1\left[x - x_1(t)\right]\ M_2\left[x - x_2(t)\right], \tag{17}$$

where

$$M_i\left[x - x_i(t)\right] = \tanh\ \left[x - x_i(t)\right] \tag{18}$$

with $i = 1\,\mathrm{or}\,2$. Substitute (17) in (16), use the expressions of the derivatives M_i' et M_i'' as functions of M_i, and perform reasonable approximations in order to find the equations for the time derivatives $\dot{x_1}$ and $\dot{x_2}$.

8. Generalize to the case of N Ising walls and show that we get

$$\dot{x_i} \approx 8\ \left[\exp\left(-2\mid x_{i+1} - x_i \mid\right) - \ \exp\left(-2\mid x_i - x_{i-1}\mid\right)\right]. \tag{19}$$

9. We consider an initial condition with a periodic array of Ising walls at a distance a. Using equation (19), perform the linear stability analysis of this state, and show that the dispersion for the growth-rate η of a mode of wave number nK is given by

$$\eta = 64\, e^{-2a} \sin^2(Ka/2). \tag{20}$$

What is the most unstable mode? How does the periodic array of Ising walls get deformed as time evolves. Give the physical interpretation of this behavior.

10. We define

$$\rho_i \equiv \exp\left(-2 \mid x_{i+1} - x_i \mid\right). \tag{21}$$

Show that $-2/log\rho_i$ represents the density of Ising walls. Find the evolution equation for $\dot{\rho}_i$. Taking the continuous limit, show that one obtains a nonlinear diffusion equation for $\rho(x,t)$. What is the expression of the diffusivity? Why does this equations show that a uniform density of walls is unstable?