Nonlinear physics

October 31, 2017 (2 hours)

Note that many questions can be solved independently

Only allowed documents: personal written notes (lectures and tutorials), lecture handouts and bilingual dictionary

I. PLOT TRAJECTORIES IN PHASE SPACE TO WARM UP

We consider the dynamical system

$$\ddot{x} = a - x^2, \tag{1}$$

where a is a real constant. We first consider the case $a > 0$.

1) Find the fixed points (or equilibrium solutions) and discuss the behavior of $x(t)$ in their vicinity.

2) Plot the potential energy as a function of x and draw the trajectories in the phase space of the system.

3) Sketch the trajectories in the phase space for $a = 0$ and $a < 0$.

II. EFFECT OF HIGH FREQUENCY FLUCTUATIONS ON A STATIONARY BIFURCATION

We consider a stationary pitchfork bifurcation

$$\dot{x} = \mu x - x^3 \tag{2}$$

1) We first assume that μ is a constant real parameter. Draw the bifurcation diagram as a function of μ with both stable and unstable solutions.

2) We now consider the case where μ is not constant because of fluctuations. We write $\mu = \mu_0 + \xi(t)$ where $\xi(t)$ represents random fluctuations with zero mean, $<\xi>= 0$ where $<\cdot>$ stands for the time average. Show that the behavior of $<x^2>$ as a function of μ_0 is not affected by the presence of the fluctuations. (Hint: divide by x and take the average).

3) We now consider a slightly more complex situation

$$\ddot{x} + \lambda\dot{x} = \mu_0 x - x^3, \tag{3}$$

where λ is a real positive constant. Perform the linear stability analysis of the solution $x = 0$ and show that for $\mu_0 \simeq 0$, there is a neutral mode and a damped mode. Give the value of μ_0 for which a bifurcation occurs. What is the name of this bifurcation?

4) We consider $\mu_0 = \hat{\mu}\epsilon$ where $\hat{\mu}$ is of order one and ϵ is small. What is the expression for the slow time scale T as a function of ϵ and t close to the bifurcation? What scaling do you expect for x versus ϵ?

5) Make a change of variable of the form $x(t) \propto y(T)$ with the appropriate scalings for x and T where $y(T)$ is of order one, and find the differential equation for $y(T)$ to leading order. What do you recover? Explain what happened.

6) We now take into account the fluctuations in (3) and therefore consider

$$\ddot{x} + \lambda\dot{x} = [\mu_0 + \xi(t)]x - x^3. \tag{4}$$

We restrict the problem to high frequency fluctuations with a large amplitude. We therefore write $\xi(t) = \frac{1}{\epsilon}\zeta(\tau)$ with $\tau = \frac{t}{\epsilon}$ and ζ of order one. Is τ a slow or fast time scale?

7) We make the change of variable $x(t) = z(t, \tau)$ in order to solve the problem using the method of multiple scales. Write the partial differential equation for $z(t, \tau)$.

8) We look for a solution in the form $z(t, \tau) = z_0(t, \tau) + \epsilon z_1(t, \tau) + \epsilon^2 z_2(t, \tau) + \cdots$ Show that to leading order we have $\frac{\partial^2 z_0}{\partial \tau^2} = 0$. Explain why we should take $z_0(t, \tau) = A(t)$ therefore discarding the other term.

9) Define the adjoint problem and show that the solvability condition consists of imposing that the inhomogeneous term has zero average over τ.

10) Is the solvability condition satisfied at order one? Give the solution for $z_1(t, \tau)$ as a function of $A(t)$ and $\chi(\tau)$ such that $\chi''(\tau) = \zeta(\tau)$.

11) Write the solvability condition at next order and find the differential equation for $A(t)$ as a function of λ, μ_0 and $< \chi'^2 >$. Show that the stationary bifurcation is delayed to positive values of μ_0 due to the presence of fluctuations.

12) We have found that without fluctuations the problem governed by equation (3) reduces to equation (2) by eliminating the damped mode in the vicinity of the bifurcation. On the other hand, a fluctuating bifurcation parameter does not change the behavior of $< x^2 >$ in (2) whereas it delays the bifurcation threshold in (3). Could you recover this result using the trick of question (2)? Why do fluctuations shift the bifurcation threshold in (3) and not in (2)?

III. COMMENSURATE-INCOMMENSURATE TRANSITIONS

We consider the problem of a crystal growing on a crystalline substrate with a different mesh size. The atoms of the growing crystal are subjected to the potential of the crystalline substrate sketched with a sinusoidal shape of wavelength a_s in figure 1. In order to minimize their interaction energy with the substrate, they tend to sit at the minima of the potential, possibly with a wavelength commensurate with the mesh size of the substrate. However, the atoms of the growing crystal also have their own interactions sketched with the springs between atoms. This potential energy is minimized for a mesh size a_0 that is in general incommensurate with a_s, i.e. a_0/a_s is not a rational number. Depending on the relative strength of these interactions, the growing crystal is either commensurate with the substrate or not. We will study this problem using an amplitude equation that we will obtain using symmetry arguments.

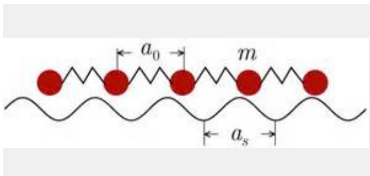

FIG. 1: The growing crystal is subjected to the potential of the substrate sketched with a sinusoidal shape of wavelength a_s. The natural (i.e. in the absence of the substrate) mesh size of the growing crystal is $a_0 \neq a_s$.

We describe the growing crystal as a system undergoing a pattern-forming stationary instability at a wavenumber $k_0 = 2\pi/a_0$. The field describing the periodic crystalline structure being

$$u(x,t) = A(X,T) \exp ik_0 x + \bar{A}(X,T) \exp -ik_0 x + \cdots , \tag{5}$$

we know that, for a stationary bifurcation, the form of the amplitude equation for A is constrained by symmetry requirements, translation invariance in space, $x \to x + x_0$ for any x_0, and reflection symmetry, $x \to -x$. The amplitude equation is

$$\frac{\partial A}{\partial T} = \mu A + \frac{\partial^2 A}{\partial X^2} - |A|^2 A. \tag{6}$$

In order to model the effect of the substrate, we consider that this stationary bifurcation occurs in the presence of a spatial forcing of wavenumber $k_s = 2\pi/a_s$. Translational invariance in space is thus broken by the presence of the substrate. The system is only invariant under discrete translations $x \to x + a_s$. This being a weaker requirement than continuous translations in space, we therefore expect that more terms are allowed in the amplitude equation that describe the formation of the growing crystal in the presence of the substrate.

We first consider the case where the growing crystal is commensurate with the substrate, i.e. $\frac{k_0}{k_s} = \frac{n}{p}$ where n and p are two integers. Therefore we write

$$u(x,t) = B(X,T) \exp i\frac{n}{p}k_s x + \bar{B}(X,T) \exp -i\frac{n}{p}k_s x + \cdots, \tag{7}$$

and look for an amplitude equation for B. We do not consider here terms differentiated in X since we look only at the mode of wavenumber $\frac{n}{p}k_s$ assumed to be equal to k_0.

1) Using the discrete translation symmetry quoted above, show that the terms allowed by the symmetry constraint are

$$\frac{\partial B}{\partial T} = \mu B + \alpha \bar{B}^{p-1} - |B|^2 B. \tag{8}$$

Why the coefficients are real?

2) We now consider that the two wavenumbers are no longer commensurate and write

$$\frac{n}{p}k_s - k_0 = \epsilon\delta, \tag{9}$$

where δ is an order one real number and ϵ is small, such that $X = \epsilon x$. $\epsilon\delta$ is called the detuning between the two wavenumbers. Use the expression of A as a function of B and the linear part of the amplitude equation for A to find the linear part of the amplitude equation for B.

3) Write down the full amplitude equation for B and give the physical interpretation of all the terms.

4) Multiply the amplitude equation for B by $\frac{\partial \bar{B}}{\partial T}$ and add the resulting expression to its complex conjugate to show that there is a Lyapunov functional

$$F[B,\bar{B}] = \int dx \left[\left| \frac{\partial B}{\partial X} \right|^2 - i\delta \left(\bar{B}\frac{\partial B}{\partial X} - B\frac{\partial \bar{B}}{\partial X} \right) - (\mu - \delta^2)|B|^2 - \frac{\alpha}{p}\left(B^p + \bar{B}^p\right) + \frac{1}{2}|B|^4 \right]. \tag{10}$$

5) We consider the case $p = 2$ and write $B = R \exp i\theta$. We can show that if $\mu - \delta^2$ is large enough, R can stay approximately constant whereas θ changes. Write the Lyapunov functional as a functional of R and θ and use this assumption to find the approximate functional of θ alone that should be minimized. Show that the corresponding Euler-Lagrange equation is

$$\frac{\partial^2 \theta}{\partial X^2} = \alpha \sin 2\theta. \tag{11}$$

6) Draw the trajectories in the phase space of equation (11) and show that there are heteroclinic orbits that connect two points with a π phase difference.

7) Show that the corresponding solutions are $\theta = 2\,\mathrm{tang}^{-1}\left(e^{\pm\sqrt{2\alpha}(x-x_0)}\right)$.

8) Show that these solitary waves in θ allows the localized relaxation of the accumulated constraints in the growing crystal due to the mismatch with the substrate.